INSTRUCTIONS: Choose five questions from below.

1. Let $E$ be a compact metric space and let $F \subset E$ be closed.
   (a) Show $E$ is complete.
   (b) If $f : E \rightarrow \mathbb{R}$ is continuous, show $f(E)$ is compact.

2. Let $a$ and $b$ be positive. Let $f_{a,b}(x) = x^a \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.
   (a) For which values of $a$ and $b$ is $f_{a,b}$ continuous?
   (b) For which values of $a$ and $b$ is $f_{a,b}$ differentiable?

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Let $x_0$ be arbitrary and for $n \geq 1$ define $x_n = f(x_{n-1})$.
   Suppose that $\lim_{n \to \infty} x_n = x$. Show that $f(x) = x$.

4. Define connected and path-connected. Show that an open connected subset $E$ of $\mathbb{R}^2$ is path-connected. Hint: If $x_0$ is an arbitrary point, let $A = \{x : \text{there is a path from } x_0 \text{ to } x \}$ and show $A = E$.

5. Suppose $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$.
   (a) Show $\lim_{n \to \infty} a_n b_n = ab$.
   (b) Define $\sigma_n = \frac{1}{n}(a_1 + \cdots + a_n)$. Show $\lim_{n \to \infty} \sigma_n = a$.

6. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded with only finitely many discontinuities. Show from the definition that $f$ is integrable.

7. Suppose that $f_n$ are continuous on $[a, b]$ and that $f_n \rightarrow f$ uniformly on $[a, b]$. Show that $\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

8. Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear. Show that $DA = A$.

9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and let $f_n(x) = f(nx)$ for $n = 1, 2, \cdots$. Assume that $\{f_n\}_{n=1}^\infty$ is equicontinuous. Show $f$ is constant.